Three-Dimensional Fluid Flow Calculations Using a Flux-Spline Method

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This paper reports the application of a flux-spline method to three-dimensional fluid flow problems. The performance of the method is compared to that of a first-order differencing scheme. The numerical results are compared with a reference solution available in the literature. The flux-spline scheme is shown to be significantly more accurate than the first-order schemes. For a prescribed accuracy, the flux-spline scheme requires a much smaller number of grid points.

Nomenclature

a	= convection-diffusion coefficient
J	= total flux of ϕ because of convection and diffusion
$\tilde{J}$	= flux-spline contribution to the total flux
J_x, J_y, J_z	= x, y, and z components of J
S	= source term
u,v,w	= velocity components in x, y, and z directions
x,y,z	= Cartesian cordinates
Γ	= diffusion coefficient
μ	= fluid viscosity
ρ	= fluid density
ϕ	= dependent variable

Subscripts

b,e,n,s,t,w,ww	= control-volume face quantities
E,P,W	= grid points (see Figs. 1 and 2)

Introduction

THE use of first-order upwind schemes for the differencing of the convective terms in the equations governing fluid flow and related phenomena introduces significant numerical error (false diffusion) in the numerical solution.[1] The presence of false diffusion makes an accurate modeling of complex recirculating flows very difficult, as it may completely mask the effects of physical diffusion. The problem of false diffusion may also cause difficulties in the evaluation of various physical models, since the resulting numerical solution does not purely reflect the outcome of the physical model embodied in the computational procedure.

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The lower-order upwind schemes are based on a one-dimensional flux balance.[1] Consequently, these schemes cannot respond to the effects of lateral transport, flow skewness, and sources. The resulting errors (false diffusion) can, in principle, be reduced to negligible levels by using a fine grid. However, the necessary degree of grid refinement is often impractical

The flux-spline scheme[4] has been applied to a variety of two-dimensional flows, both linear and nonlinear, and has been shown to be superior to the lower-order schemes. The purpose of this paper is to demonstrate the accuracy of the flux-spline scheme for three-dimensional recirculating flows. For this purpose, laminar flow in a lid-driven cubic cavity has been computed.

The results from the flux-spline scheme have been compared with an accurate numerical solution available in the literature. For reference, results from the lower-order power-law differencing scheme[1] have also been included.

Mathematical Formulation

The calculation procedure used in this study is based on the primitive variable formulation of the Navier-Stokes equations. The conservation equations are discretized using a control-volume approach. The procedure is described in detail in Ref. 1.

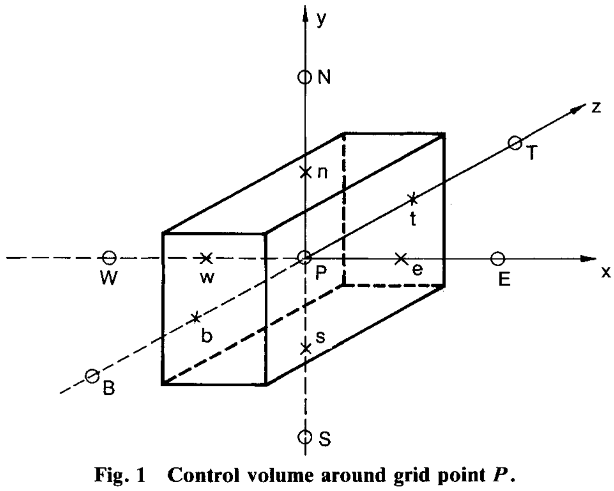

Fig. 1 Control volume around grid point *P*.

The conservation equations for all dependent variables may be expressed in the following general form:

$$\frac{\partial}{\partial x} J_x + \frac{\partial}{\partial y} J_y + \frac{\partial}{\partial z} J_z = S \tag{1}$$

where J_x, J_y, and J_z are the total (convection and diffusion) fluxes, defined by

$$J_x = \rho u \phi - \Gamma \frac{\partial \phi}{\partial x} \tag{2a}$$

$$J_y = \rho v \phi - \Gamma \frac{\partial \phi}{\partial y} \tag{2b}$$

$$J_z = \rho w \phi - \Gamma \frac{\partial \phi}{\partial z} \tag{2c}$$

The integration of Eq. (1) over the control volume surrounding the grid point P (Fig. 1) gives

$$(J_{x,e} - J_{x,w})\Delta y \Delta z + (J_{y,n} - J_{y,s})\Delta x \Delta z + (J_{z,t} - J_{z,b})\Delta x \Delta y = S \Delta x \Delta y \Delta z \tag{3}$$

A discretization scheme is needed to relate the flux at each control-volume face to the values of the dependent variable at the neighboring grid points. The results presented in this paper have been obtained using the power-law differencing scheme and flux-spline scheme. A brief description of these schemes is presented next.

Power-Law Differencing Scheme

This scheme is based on a curve fit to the exact solution of the one-dimensional convection-diffusion equation without a source. Since this formulation is based on a purely one-dimensional flux balance, it leads to significant numerical errors in the presence either of strong source terms, or of crossflow gradients in multidimensional flows coupled with the grid-to-flow skewness. The flux-spline scheme includes these effects in the interpolation profile between the grid points.

Flux-Spline Scheme

The flux-spline scheme considered here is based on the assumption that within a control volume the total flux in a given direction varies linearly along the coordinate direction. For example, the flux in the x direction for the control volume around the grid point P (Fig. 2) is given by

$$J_x = \rho u \phi - \Gamma \frac{\partial \phi}{\partial x} = J_{x,w} + \frac{J_{x,e} - J_{x,w}}{\Delta x} x \tag{4}$$

The integration of Eq. (4) leads to the following expression for the variation of ϕ in the x direction

$$\phi = a + b \exp(\rho u x / \Gamma) + cx \tag{5}$$

where the constants a, b, and c for a given control volume can be expressed in terms of $J_{x,e}$, $J_{x,w}$, and ϕ_P.

Equation (5) gives the variation of ϕ within a control volume. For two adjacent control volumes, the ϕ profiles are such that they imply the same total flux at the common interface. In addition, these profiles must also give a unique value of ϕ at the common interface. This continuity-of-ϕ (spline continuity) condition for the interface between grid points W and P can be expressed as

$$J_{x,w} = (D_{x,w}\phi_W - E_{x,w}\phi_P) + B_{x,w}(J_{x,w} - J_{x,e}) + C_{x,w}(J_{x,w} - J_{x,ww}) \tag{6}$$

Here the expression $(D_{x,w}\phi_W - E_{x,w}\phi_P)$ is identical to that obtained from the lower-order exponential scheme (e.g., Ref. 1) that is based on the assumption that the total flux is uniform within a control volume. The extra terms involving B_x and C_x result from the linear variation of flux. For ease of presentation, Eq. (6) is rewritten as

$$J_{x,w} = (D_{x,w}\phi_W - E_{x,w}\phi_P) + \tilde{J}_{x,w} \tag{7}$$

It should be noted that the additional terms, such as $\tilde{J}_{x,w}$, involve the differences in flux values at adjacent faces of the control volume. That there is a difference in flux indicates the presence of a source term and/or multidimensionality. (A change of flux in one direction is felt as a source term in another direction.)

Similar expressions can also be derived for fluxes in other coordinate directions. Substituting these expressions in Eq. (3) and utilizing the discrete form of the continuity equation, the following discretization equation for ϕ is obtained:

$$a_P \phi_P = \Sigma a_{nb} \phi_{nb} + b + \hat{S} \tag{8}$$

The values of the influence coefficients a_{nb} are identical to the coefficients obtained from the exponential scheme. The contribution of the flux-spline formulation is contained in the term $\hat{S}$, which is given by

$$\hat{S} = (\tilde{J}_{x,w} - \tilde{J}_{x,e})\Delta y \Delta z + (\tilde{J}_{y,s} - \tilde{J}_{y,n})\Delta x \Delta z + (\tilde{J}_{z,b} - \tilde{J}_{z,t})\Delta x \Delta y \tag{9}$$

A three-dimensional situation is governed by four field variables: ϕ, J_x, J_y, and J_z. The four sets of equations that determine these variables are 1) the conservation equation for ϕ; 2) the spline-continuity condition in the x direction; 3) the spline-continuity condition in the y direction; and 4) the spline-continuity condition in the z direction.

The solution of these equations is obtained in an iterative manner. In the beginning, $\tilde{J}_x$, $\tilde{J}_y$, and $\tilde{J}_z$ are set equal to zero; then the conservation equation for ϕ reduces to the lower-order formulation and can be easily solved. The solution leads to new estimates for the fluxes J_x, J_y, and J_z from which new $\tilde{J}_x$, $\tilde{J}_y$, and $\tilde{J}_z$ can be calculated. The ϕ equation is now solved with the flux-spline contribution to the source term. This process is repeated until convergence is achieved.

The coupling between the continuity and momentum equations is handled using the SIMPLER algorithm.[1] The steps involved are identical to those for the lower-order formulation (e.g., Ref. 1). Complete details of the flux-spline scheme are available in Refs. 4 and 9.

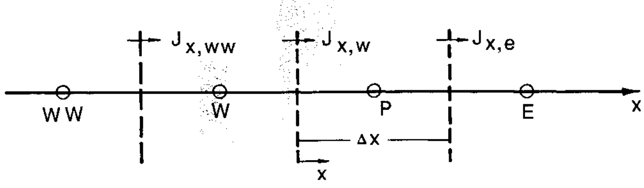

Fig. 2 One-dimensional situation.

Results and Discussion

To assess the performance of the flux-spline scheme, computations have been made for laminar flow in a lid-driven cubic cavity.

The physical situation considered here is shown in Fig. 3. The governing equations have been solved for half of the cubic box, since the flow is symmetric about the $z = 0.5$ plane. The boundary conditions for this problem are as follows. The velocity components are specified on all walls. At the symmetry plane, the normal component of velocity and the normal gradients of other velocity components are zero. For a given geo-

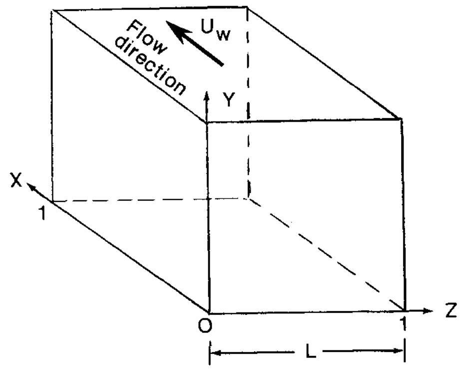

Fig. 3 Cubic cavity flow configuration.

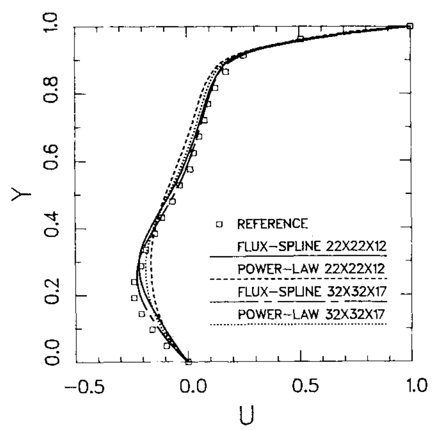

Fig. 4 The U-velocity profile on the vertical centerline of z = 0.5.

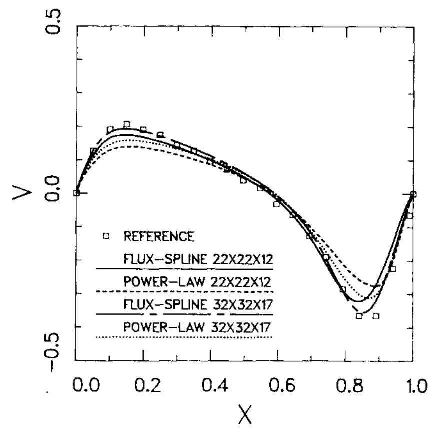

Fig. 5 The V-velocity profile on the horizontal centerline of $z = 0.5$ plane.

metrical configuration, the flow is characterized by the Reynolds number ($Re = \rho U_w L/\mu$). Here results have been obtained for a Reynolds number of 4×10^2.

This problem was solved using two grids—$22 \times 22 \times 12$ (x,y,z) and $32 \times 32 \times 17$. The grid spacing was uniform for both grids. The results of present calculations are compared with the numerical solution of Ku et al.[10] obtained using a pseudospectral method ($25 \times 25 \times 13$ mode). For a two-dimensional problem at $Re = 4 \times 10^2$, the 25×25 solution of Ku et al. compares well with the grid-independent solution of Ghia et al.,[11] who employed the second-order upwind differencing scheme on a 129×129 finite-difference grid. Since the flow considered is laminar, there are no uncertainties regarding mathematical models and a grid-independent numerical solution is an accurate discrete representation of the exact solution. The solution of Ku et al. has been taken as the *reference* in this paper. The performance of flux-spline and power-law schemes is evaluated by comparing their results on different grids against this reference solution. The accuracy of a numerical solution is judged by its closeness to the reference solution.

Figures 4 and 5 show the profiles of u component of velocity on the vertical centerline and v component on the horizontal centerline of the symmetry plane $z = 0.5$. For a given grid, the flux-spline results are closer to the reference solution than the power-law results. In particular, the flux-spline scheme preserves the regions of steep velocity gradients; the use of the power-law scheme leads to smeared profiles indicating the presence of excessive numerical diffusion. It is noted that the coarse grid ($22 \times 22 \times 12$) flux-spline solution is more accurate than the fine grid ($32 \times 32 \times 17$) power-law solution. This indicates that, for a prescribed accuracy, the flux-spline scheme requires much fewer number of grid points than the power-law scheme. Further, the accuracy of the flux-spline solution on the $32 \times 32 \times 17$ grid is comparable to that of the reference solution obtained on a much finer grid. Thus, to the extent that the reference solution is correct, these comparisons establish the higher accuracy of the flux-spline scheme. The preceding calculations presented above were started from zero velocity and pressure fields. The calculations were terminated when the summed normalized residual in each equation was below 10^{-4}. During the coefficient update, an underrelaxation factor of 0.7 was used for velocities; no underrelaxation is required for pressure. For the $32 \times 32 \times 17$ grid, the number of iterations required by the power-law and flux-spline schemes were 125 and 145, respectively. The corresponding execution times (CRAY-XMP) are 255 and 320 s respectively.

Concluding Remarks

In this paper, the flux-spline method has been applied to a three-dimensional fluid flow problem. For a fixed number of grid points, this scheme has been shown to be significantly more accurate than the lower-order schemes. The use of the flux-spline scheme leads to an accurate prediction of the complex flowfield even on a rather coarse grid.

Acknowledgments

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Several goals have been established for *JA* beginning in FY'91:

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B.M.M.